#1 $\vec{F}(x,y) = \left(2+y+e^{x}\sin y\right)\hat{i} + \left(4+x+e^{x}\cos y\right)\hat{j}$

$\vec{\nabla} f = \vec{F}$

$\langle f_x, f_y \rangle = \langle 2+y+e^{x}\sin y,\ 4+x+e^{x}\cos y \rangle$

$f_x = 2+y+e^{x}\sin y \longrightarrow f(x,y) = \int 2+y+e^{x}\sin y\, dx$

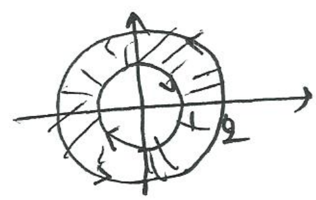
$= 2x + yx + e^{x}\sin y$

$f_y = 4+x+e^{x}\cos y \longrightarrow f(x,y) = \int 4+x+e^{x}\cos y\, dy$

$= 4y + xy + e\sin y$

$$\boxed{f(x,y) = 2x + yx + e^{x}\sin y + 4y}$$

#2 $\int_C (\sin \sqrt[3]{x})\, dx + (x^3+3xy^2)\, dy$



C is closed ✓

simple ✓

positively oriented ✓

$\therefore$ Green's Theorem

$\int_C \vec{F}\cdot d\vec{r} = \iint_D Q_x - P_y\, dA$

$= \iint_D 3x^2+3y^2\, dA$

$P = \sin\sqrt[3]{x} \rightarrow P_y = 0$

$Q = x^3+3xy^2 \rightarrow Q_x = 3x^2+3y^2$

$= \int_0^{2\pi}\int_1^2 3r^2\, r\, dr\, d\theta$

$= \int_0^{2\pi} 3\frac{r^4}{4}\Big|_1^2 d\theta$

$= (2\pi)\left(\frac{3}{4}\right)(2^4-1^4)$

$$\boxed{= \frac{45\pi}{2}}$$

#3 $\vec{F}(x,y) = \langle xy^2, x+y \rangle$

① Is $\vec{F}$ Cons?

$P = xy^2 \longrightarrow P_y = 2xy$

$Q = x+y \longrightarrow Q_x = 1$

$\neq$ NO

③ Is the line segment closed? NO ∴ directly computation means we have to parametrize the line segment

$C: (2,4) \longrightarrow (2,1)$

$$\begin{cases} x = 2 \\ y = t \end{cases}$$

$1 \le t \le 4$

$t=1$ $(2,1)$ $\xrightarrow{-C}$ $t=4$ $(2,4)$

$$\int_C \vec{F}\cdot d\vec{r} = -\int_{-C} \vec{F}\cdot d\vec{r}$$

$$= -\int_{-C} \langle xy^2, x+y \rangle \langle dx, dy \rangle$$

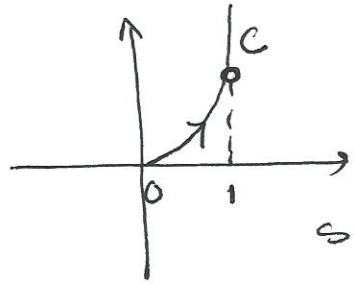
$$= -\int_{-C} xy^2\,dx + (x+y)\,dy$$

$$= -\int_1^4 \cancelto{0}{2(t)^2(0)} + (2+t)(1)\,dt$$

$$= -\left[2t + \frac{t^2}{2}\right]_1^4$$

$$= -\left(8 + \frac{4}{2} - 2 - \frac{1}{2}\right)$$

4



$ds \to |\vec{r}'(t)|\,dt$

step#1 parametrize the curve

$$\vec{r}(t): \begin{cases} x = t \\ y = t^2 \end{cases}$$

$0 \leq t \leq 1$

step#2 Jacobian

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{1^2 + (2t)^2}$$

$$= \sqrt{1+4t^2}$$

step#3

$$\int_C \frac{y^2}{x^3}\,ds = \int_0^1 \frac{(t^2)^2}{t^3}\sqrt{1+4t^2}\,dt$$

$$= \int_0^1 t\sqrt{\underbrace{1+4t^2}_{u}}\,dt$$

$u = 1+4t^2$
$du = 8t\,dt$

$t=0 \to u=1$
$t=1 \to u=5$

$$= \frac{1}{8}\int_1^5 \sqrt{u}\,du$$

$$= \frac{1}{8}\left[\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^5$$

$$= \frac{1}{\cancel{8}_4}\,\frac{\cancel{2}}{3}\left(5^{\frac{3}{2}} - 1^{\frac{3}{2}}\right)$$

$5^{\frac{3}{2}} = \sqrt{5^3} = 5\sqrt{5}$

$$\boxed{= \frac{1}{12}\left(5\sqrt{5}-1\right)}$$

5

$f(x,y) = xy\cos y$

$\vec{F}(x,y) = \langle y\cos y,\ x\cos y - xy\sin y \rangle$

$\nabla f = \vec{F}$ $\therefore$ f is potential function of Conservative vector Feild $\vec{F}$ $\therefore$ Apply FTLI

$$\int_C \vec{F}\cdot d\vec{r} = f(\text{end}) - f(\text{start}) = xy\cos y \Big|_{(\pi^2,\pi)} - xy\cos y \Big|_{(0,0)}$$

$$= \pi\,\pi^2\cos\pi - 0$$

$$\boxed{= -\pi^3}$$

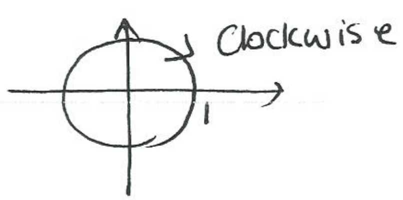
$\vec{r}(t) = \langle t^2, t \rangle \quad 0 \le t \le \pi$

start: $t=0$ $\quad \vec{r}(0) = \langle 0, 0 \rangle$

end $t=\pi$ $\quad \vec{r}(\pi) = \langle \pi^2, \pi \rangle$

6

$\int_C \vec{F}\cdot d\vec{r}$; $\vec{F}(x,y) = \langle e^{2x} + x^2 y,\ e^{2y} - xy^2 \rangle$



Closed ✓
Simple ✓
positively oriented ✗ NO

$P = e^{2x} + x^2 y \rightarrow P_y = x^2$
$Q = e^{2y} - xy^2 \rightarrow Q_x = -y^2$
$\neq \therefore \vec{F}$ is NOT Conservative

$$\int_C \vec{F}\cdot d\vec{r} = \iint_D Q_x - P_y\, dA \overset{\text{negatively-oriented}}{=} -\iint_D -y^2 - x^2\, dA$$

$$= \iint_D y^2 + x^2\, dA$$

$$= \int_0^{2\pi}\int_0^1 r^2\, r\, dr\, d\theta = \int_0^{2\pi} \frac{r^4}{4}\Big|_0^1 d\theta = \frac{1}{4}(2\pi) = \boxed{\frac{\pi}{2}}$$

8 $\vec{F}(x,y,z) = \langle \sin xy,\ y+z,\ x-yz \rangle$

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \langle \sin xy,\ y+z,\ x-yz \rangle$$

$$= \frac{\partial \sin xy}{\partial x} + \frac{\partial}{\partial y}(y+z) + \frac{\partial}{\partial z}(x-yz)$$

$$= y\cos xy + 1 + (-y) = y(\cos xy - 1) + 1$$

$$\operatorname{Curl}\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin xy & y+z & x-yz \end{vmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}(x-yz) - \frac{\partial}{\partial z}(y+z),\ \frac{\partial}{\partial z}(\sin xy) - \frac{\partial}{\partial x}(x-yz),\ \frac{\partial}{\partial x}(y+z) - \frac{\partial}{\partial y}\sin xy \right\rangle$$

$$= \langle -z-1,\ -1,\ x\cos xy \rangle$$

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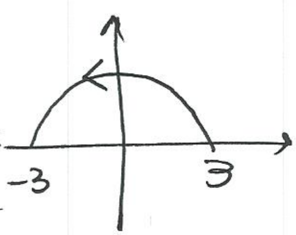
9 $f(x,y) = x^2 y$

step#1 parametrize the curve



$$\begin{cases} x = 3\cos t \rightarrow dx = -3\sin t\, dt \\ y = 3\sin t \rightarrow dy = 3\cos t\, dt \end{cases}$$

$0 \le t \le \pi$

step#2 Jacobian $= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$$= \sqrt{(-3\sin t)^2 + (3\cos t)^2}$$

$$= \sqrt{3^2(\sin^2 t + \cos^2 t)}$$

$$\boxed{= 3}$$

step#3 $\int_C x^2 y\, ds = \int_0^{\pi} (3\cos t)^2 (3\sin t)(3)\, dt = 81\int_0^{\pi} \cos^2 t \sin t\, dt$

$$= 81\left[\frac{\cos^3 t}{3}\right]_0^{\pi}$$

$$= 81\left[-\frac{1}{3} - \frac{1}{3}\right]$$

$$\boxed{= -27}$$

10 Let $\vec{F} = \langle 1+y, x-\sqrt{y} \rangle$

a) $\vec{F}$ is Cons?

$P = 1+y \rightarrow P_y = 1$

$Q = x-\sqrt{y} \rightarrow Q_x = 1$

$\checkmark \Rightarrow F$ is Cons.

b) potential function

$\vec{\nabla f} = \vec{F}$

$\langle f_x, f_y \rangle = \langle 1+y, x-\sqrt{y} \rangle$

$f_x = 1+y \longrightarrow f(x,y) = \int 1+y \, dx$

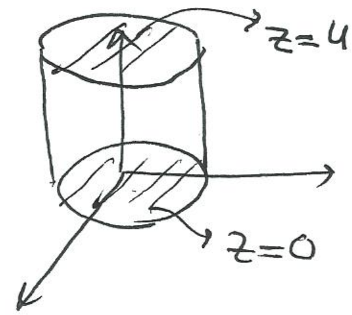
$= x + yx$

$f_y = x - \sqrt{y} \longrightarrow f(x,y) = \int x - \sqrt{y} \, dy$

$= xy - \frac{2y}{3}^{3/2}$

$$\boxed{f(x,y) = xy + x - \frac{2}{3} y^{3/2}}$$

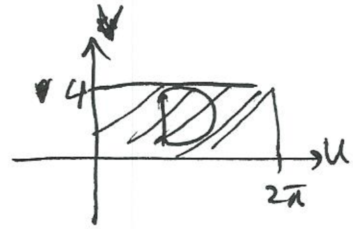
#11 $x^2+y^2=4$ Find surface area between $z=0$, $z=4$



$$A(S)=\iint_D |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

#1 parametrize the surface $x^2+y^2=4$

$$\begin{cases} x = 2\cos u \\ y = 2\sin u \\ z = v \\ 0 \le u \le 2\pi,\ 0 \le v \le 4 \end{cases}$$



#2 Jacobian $= |\vec{r}_u \times \vec{r}_v| = 2$

#3 $$A(S) = \iint_D 2\, dA$$

$$= 2\ \text{area}(D)$$

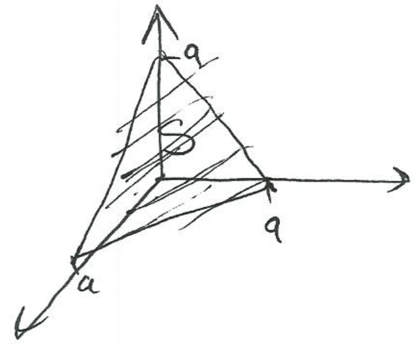
$$= 2(2\pi)(4)$$

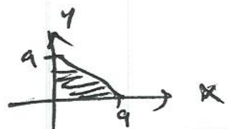
$$\boxed{= 16\pi}$$

12 $\quad$ "Flux out" $= \iint_S \vec{F}\cdot\hat{n}\,dS$

#1 parametrize

$$\begin{cases} x = x \\ y = y \\ z = 9 - x - y \\ (x,y) \in D \end{cases}$$





#2 $\quad$ Jacobian $= |\vec{r}_x \times \vec{r}_y| = \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{(-1)^2 + (-1)^2 + 1} = \sqrt{3}$

#3 $\quad \hat{n} = \dfrac{\vec{r}_x \times \vec{r}_y}{|\vec{r}_x \times \vec{r}_y|} = \dfrac{\langle -f_x, -f_y, 1\rangle}{\sqrt{f_x^2 + f_y^2 + 1}} = \dfrac{\langle 1, 1, 1\rangle}{\sqrt{3}}$

#4 $\quad \iint_S \vec{F}\cdot\hat{n}\,dS = \iint_D \langle xze^y, -xze^y, 9-x-y\rangle \cdot \dfrac{\langle 1, 1, 1\rangle}{\cancel{\sqrt{3}}}\,\cancel{\sqrt{3}}\,dA$

$$= \iint_D 9 - x - y \; dA$$

$$= \int_0^9 \int_0^{9-x} 9 - x - y \; dy\,dx$$