7)

Consider the following proposition: For each integer a, $a \equiv 2 \pmod 8$ if and only if $(a^2 + 4a) \equiv 4 \pmod 8$.

a.

Write the proposition as the conjunction of two conditional statements.

$(\forall a \in \mathbb{Z})\ ([(a \equiv 2 \pmod 8) \rightarrow ((a^2 + 4a) \equiv 4 \pmod 8)] \wedge [((a^2 + 4a) \equiv 4 \pmod 8) \rightarrow (a \equiv 2 \pmod 8)])$

b.

Determine if the two conditional statements in Part (a) are true or false. If a conditional statement is true, write a proof, and if it is false, provide a counterexample.

P → Q is True

We are given $a \equiv 2 \pmod 8$ and $(a^2 + 4a) \equiv 4 \pmod 8$

We want to show that $a \equiv 2 \pmod 8 \rightarrow (a^2 + 4a) \equiv 4 \pmod 8$

Since $a \equiv 2 \pmod 8$, then there exists $k \in \mathbb{Z}$ such that

$$a - 2 = 8k \text{ or } a = 8k + 2$$

Then,

$$a^2 + 4a - 4 = (8k + 2)^2 + 4(8k + 2) - 4$$
$$= 64k^2 + 32k + 4 + 32k + 8 - 4$$
$$= 64k^2 + 64k + 8$$
$$= 8\ (8k^2 + 8k + 1)$$

Since $8k^2 + 8k + 1$ is is closed by multiplication and addition, $8k^2 + 8k + 1 = m$ for some integer m

Thus,

$$(a^2 + 4a) \equiv m \pmod 8$$

Or

$$(a^2 + 4a) \equiv 4 \pmod 8$$

Therefore,

$$a \equiv 2 \pmod 8 \rightarrow (a^2 + 4a) \equiv 4 \pmod 8$$

Q → P is False

We want to show that $(a^2 + 4a) \equiv 4 \pmod 8 \rightarrow a \equiv 2 \pmod 8$

If we let $a = 14$, then we have

$[(14^2 + 4(14)) = 252] \equiv 4 \pmod{8}$

But, $14 \not\equiv 2 \pmod{8}$

c. Is the given proposition true or false? Explain.

The given proposition is false since $Q \rightarrow P$ is false.