

Exam 3 Solutions

1. (10 points) Prove Corollary 4.30: Every polynomial $f(x)$ of odd degree in $\mathbb{R}[x]$ has a root in \mathbb{R} .

Proof: Suppose $f(x)$ is an arbitrary polynomial in $\mathbb{R}[x]$ with odd degree. We need to show that $f(x)$ has a root in \mathbb{R} . Suppose $f(x)$ has degree n . By Corollary 4.27, we can write $f(x)$ as

$$f(x) = c(x - a_1)(x - a_2) \cdots (x - a_n)$$

for some $c, a_1, a_2, \dots, a_n \in \mathbb{C}$. Then $f(x)$ has n roots, counting multiplicities. By Lemma 4.28, the complex, nonreal roots must come in complex conjugate pairs. Therefore the number of complex, nonreal roots must be even. Since the total number of roots is odd, then $f(x)$ must have a real root.

2. (18 points) Show that a 40° angle is not constructible. Use this to show that a regular nonagon (9-sided polygon) is not constructible.

To show that a 40° angle is not constructible, it suffices to show that $\cos 40^\circ$ is not constructible. If we substitute $t = 40^\circ$ into the trigonometric identity $\cos 3t = 4 \cos^3 t - 3 \cos t$, we obtain

$$\begin{aligned}\cos 120^\circ &= 4 \cos^3 40^\circ - 3 \cos 40^\circ \\ -\frac{1}{2} &= 4 \cos^3 40^\circ - 3 \cos 40^\circ\end{aligned}$$

Then

$$8 \cos^3 40^\circ - 6 \cos 40^\circ + 1 = 0.$$

This shows that $\cos 40^\circ$ is a root of the polynomial $f(x) = 8x^3 - 6x + 1$. By the Rational Root Test, if r/s is a rational root of $f(x)$ in lowest terms, then $r|1$ and $s|8$. So the possible rational roots are

$$r/s = \pm \frac{1}{8}, \pm \frac{1}{4}, \pm \frac{1}{2}, \pm 1.$$

It's straightforward to check that none of these are a root of $f(x)$, so $f(x)$ has no rational roots.

Theorem 15.9 then implies that $f(x)$ has no constructible numbers as roots. Since $\cos 40^\circ$ is a root, then $\cos 40^\circ$ cannot be constructible. Therefore a 40° degree angle is not constructible.

To show that a regular nonagon is not constructible, use a proof by contradiction. Suppose we can construct a regular nonagon. The center of the nonagon can be located by taking the intersection of two lines joining vertices to the midpoints of the opposite sides. We can then draw lines from the center to two adjacent vertices. These lines will form a 40° angle at the center. Then this enables us to construct a 40° angle, a contradiction. Thus a regular nonagon is not constructible.

3. (40 points) Let $G = \mathbb{Z}$, the set of integers, under the operation

$$a * b = a + b + 1 \quad \text{for all } a, b \in G.$$

It can be shown that G is a group under this operation with identity element $e_G = -1$.

(a) Verify group axiom G4 Existence of Inverse Elements for G .

We need to show

$$(\forall a \in G) (\exists a^{-1} \in G) [a * a^{-1} = e \wedge a^{-1} * a = e].$$

Proof: Suppose $a \in G$ is arbitrary. We need to show there exists an inverse element $a^{-1} \in G$.

Find an a^{-1} .

The element a^{-1} must satisfy

$$a * a^{-1} = e_G$$

$$a + a^{-1} + 1 = -1$$

$$a^{-1} = -a - 2$$

Choose $a^{-1} = -a - 2$.

Verify that a^{-1} satisfies:

(i) Domain: $a^{-1} \in G$

(ii) Propositional Function:

$$a * a^{-1} = e_G \wedge a^{-1} * a = e_G$$

Since $a \in G$, then a is an integer. Since the integers are closed under addition and subtraction, then $a^{-1} = -a - 2$ is also an integer so $a^{-1} \in G$. To verify the propositional function,

$$\begin{aligned} a * a^{-1} &= a * (-a - 2) \\ &= a + (-a - 2) + 1 \\ &= -1 = e_G \end{aligned}$$

$$\begin{aligned} a^{-1} * a &= (-a - 2) * a \\ &= (-a - 2) + a + 1 \\ &= -1 = e_G \end{aligned}$$

Therefore $a^{-1} = -a - 2$ is an inverse element for a and axiom G4 holds.

(b) Show that G is cyclic with generator $a = 0$.

We need to show that

$$G = \langle 0 \rangle = \{0^n \mid n \in \mathbb{Z}\}.$$

That is, we need to show that each integer can be written as some power of 0. We have

$$0^1 = 0$$

$$0^2 = 0 * 0 = 0 + 0 + 1 = 1$$

$$0^3 = 0 * 0^2 = 0 * 1 = 0 + 1 + 1 = 2$$

$$0^4 = 0 * 0^3 = 0 * 2 = 0 + 2 + 1 = 3$$

\vdots

and

$$0^0 = e_G = -1$$

$$0^{-1} = -0 - 2 = -2$$

$$0^{-2} = 0^{-1} * 0^{-1} = (-2) * (-2) = (-2) + (-2) + 1 = -3$$

$$0^{-3} = 0^{-1} * 0^{-2} = (-2) * (-3) = (-2) + (-3) + 1 = -4$$

\vdots

We can see that each integer is a power of 0. Thus $G = \langle 0 \rangle$ so G is cyclic with generator 0.

3. (continued)

(c) Show that the set H of odd integers is a subgroup of G .

Proof: H is clearly nonempty. To show H is a subgroup, we need to show that it satisfies properties SG1 and SG2 of Theorem 7.10.

SG1. Closure. For all $a, b \in H$, $a * b \in H$.

Suppose a and b are arbitrary elements of H . That is, a and b are arbitrary odd integers. We need to show $a * b \in H$. To show this, we need to show that $a * b$ is an odd integer. By the definition of $*$,

$$a * b = a + b + 1.$$

Since a and b are both odd and 1 is odd, then $a + b + 1$ is an odd integer. Therefore $a * b \in H$ so H is closed.

SG2. Containment of Inverses. For all $a \in H$, $a^{-1} \in H$.

Suppose a is an arbitrary element of H . That is, a is an arbitrary odd integer. We need to show that $a^{-1} \in H$. To show this, we need to show that a^{-1} is an odd integer. The inverse of a is

$$a^{-1} = -a - 2.$$

Since a is an odd integer, then $-a - 2$ is an odd integer. Therefore $a^{-1} \in H$, as required. Since K is closed and contains inverses, then it is a subgroup.

Therefore H is a subgroup of G .

4. (24 points) Suppose G and H are groups and $f : G \rightarrow H$ is a homomorphism. Define the set

$$K = \{a \in G \mid f(a) = e_H\}.$$

Prove that K is nonempty and is a subgroup of G .

Proof: Since f is a homomorphism, then $f(e_G) = e_H$, which implies $e_G \in K$. Therefore K is nonempty. To show K is a subgroup, we need to show that it satisfies properties SG1 and SG2 of Theorem 7.10.

SG1. Closure. For all $a, b \in K$, $ab \in K$.

Suppose a and b are arbitrary elements of K . We need to show $ab \in K$. To show this, we need to show that

$$f(ab) = e_H.$$

Since $a, b \in K$, the $f(a) = e_H$ and $f(b) = e_H$. Since f is a homomorphism, then

$$f(ab) = f(a)f(b) = e_H e_H = e_H.$$

Therefore $ab \in K$ so K is closed.

SG2. Containment of Inverses. For all $a \in K$, $a^{-1} \in K$.

Suppose a is an arbitrary element of K . We need to show that $a^{-1} \in K$. To show this, we need to show that

$$f(a^{-1}) = e_H.$$

Since $a \in K$, then $f(a) = e_H$. Since f is a homomorphism, then Theorem 7.19(2) implies

$$f(a^{-1}) = (f(a))^{-1} = (e_H)^{-1} = e_H.$$

Therefore $a^{-1} \in K$, as required. Since K is closed and contains inverses, then it is a subgroup.

5. (16 points) Let G be a group and suppose $a \in G$ has finite order n . Prove: If $a^k = e$, then $n \mid k$.

Proof: Use a proof by contradiction. Suppose $a^k = e$ but $n \nmid k$. We then need to derive a contradiction. By the Division Algorithm, there exist integers q and r such that

$$k = qn + r \quad \text{and} \quad 0 \leq r < n.$$

Since $n \nmid k$, then the remainder $r \neq 0$, which means $r > 0$. Since $n = qn + r$, then

$$a^k = a^{qn+r} = (a^n)^q a^r$$

Since a has order n , then $a^n = e_G$. Then

$$a^k = (a^n)^q a^r = e^q a^r = a^r$$

Since $a^k = e$, then this implies $a^r = e$. Since $|a| = n$, then n is the smallest positive integer such that $a^n = e$. Since $0 < r < n$, this implies $a^r \neq e_G$. This gives a contradiction. Thus the result is true.